

On matching LTB and Vaidya spacetimes through a null hypersurface

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Abstract

In this work the matching of a LTB interior solution representing dust matter to the Vaidya exterior solution describing null fluid through a null hypersurface is studied. Different cases in which one is able to smoothly match these two solutions to Einstein equations along a null hypersurface are discussed.

1 Introduction

Matching of two exact solutions to the Einstein's equations along a null hypersurface, although well formulated now [1], has not had many applications yet. One already knows from the original Penrose work [2] how interesting such matchings can be: a simple matching of two Minkowski spacetimes may lead to gravitational impulsive or shock waves across the null boundary. There are, however, not many examples of such matchings within general relativity (see [3] and references therein).

Lemos [4] and Hellaby [5] have shown that the Vaidya spacetime which describes incoherent null radiation moving along null geodesics, can be obtained mathematically from the Lemaitre-Tolman-Bondi (LTB) spacetime, which represents an inhomogeneous distribution of dust fluid following timelike geodesics, by taking the limit of the function $E(r) \rightarrow \infty$ in the LTB solution (see Eq. (1) below). In this case, the two metrics represent the same collapse and their naked singularities are of the same nature. Recently, Gao and Lemos [6] studied the possibility of matching these two solutions in one single spacetime through a null hypersurface. In this way, by construction of a continuous coordinate system which is comoving with both the LTB and Vaidya observers they showed that the requirement for a smooth matching implies the divergence of $E(r)$ at the place of the null hypersurface. The authors also demonstrated that the

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spacetime metric can be at least C^1 and the energy- momentum tensor is continuous across the null hypersurface.

In this note we use the Barrabès-Israel (BI) null shell formalism [1] to investigate the smooth matching for the configuration studied in Ref. [6] and to find the matching conditions.

Conventions. Natural geometrized units, in which $G = c = 1$ are used throughout the paper. The null hypersurface is denoted by Σ . The symbol $|_\Sigma$ means "evaluated on the null hypersurface". We use square brackets $[F]$ to denote the jump of any quantity F across Σ . Latin indices range over the intrinsic coordinates of Σ denoted by ξ^a , and Greek indices over the coordinates of the 4-manifolds.

2 Matching Conditions

We choose the LTB metric to be written in the synchronous comoving coordinates in the form [7]

$$ds_-^2 = -dt^2 + \frac{R'^2}{1 + E(r)} dr^2 + R^2(t, r)(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (1)$$

where the overdot and prime denote partial differentiation with respect to t and r , respectively, and $E(r)$ is an arbitrary real function of r such that $E(r) > -1$. Then the corresponding Einstein field equations turn out to be

$$\dot{R}^2(t, r) = E(r) + \frac{2M(r)}{R}, \quad (2)$$

$$4\pi\rho_L(t, r) = \frac{M'(r)}{R^2 R'}, \quad (3)$$

where ρ_L is the energy density and $M(r)$ is another arbitrary function interpreted as the effective gravitational mass within r . We then take an incoming Vaidya spacetime described by the following metric [8]

$$ds_+^2 = -\left(1 - \frac{2M(v)}{R}\right) dv^2 + 2dv dR + R^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (4)$$

where $M(v) > 0$ is an arbitrary function of the ingoing null coordinate v , representing the mass accreted at time v . The energy momentum tensor for the metric (4) is of pure radiation type

$$T_{\mu\nu} = \frac{1}{4\pi R^2} \frac{dM(v)}{dv} l_\mu l_\nu, \quad l_\mu = -\delta_\mu^v, \quad l_\mu l^\mu = 0, \quad (5)$$

where one can show the identification $dM(v)/dv = 4\pi R^2 \rho_R$, with ρ_R denotes the radiation density. To glue the interior LTB inhomogeneous region to the exterior Vaidya spacetime along the null hypersurface Σ we need to have

$$r = r(t), \quad \frac{dr}{dt} = \frac{\sqrt{1 + E}}{R'(t, r)}|_\Sigma, \quad (6)$$

in the minus coordinates, while in the plus coordinates, the hypersurfaces $v = \text{constant}$ turn out to be null. Now, the requirement for the continuity of the induced metric on Σ yields the following matching condition

$$R(t, r) \stackrel{\Sigma}{=} R, \quad (7)$$

where $\overset{\Sigma}{\equiv}$ means that the equality must be evaluated on Σ . For further applications, we note that differentiation of (7) on Σ leads to

$$\dot{R}dt + R'dr \overset{\Sigma}{=} dR. \quad (8)$$

Taking $\xi^a = (\lambda, \theta, \varphi)$ with $a = 1, 2, 3$ as the intrinsic coordinates on Σ while identifying $-R$ with the parameter λ on the null generators of the hypersurface we calculate the tangent basis vectors $e_a = \partial/\partial\xi^a$ on both sides of Σ . Using Eqs. (6) and (8), we get

$$e_\lambda^\mu|_+ = (0, -1, 0, 0)|_\Sigma, \quad e_\theta^\mu|_+ = \delta_\theta^\mu, \quad e_\varphi^\mu|_+ = \delta_\varphi^\mu, \quad (9)$$

$$e_\lambda^\mu|_- = \frac{-1}{\dot{R} + \sqrt{1+E}} \left(1, \frac{\sqrt{1+E}}{R'}, 0, 0 \right)|_\Sigma, \quad e_\theta^\mu|_- = \delta_\theta^\mu, \quad e_\varphi^\mu|_- = \delta_\varphi^\mu. \quad (10)$$

Choosing the tangent-normal vector n^μ to coincide with the tangent basis vector associated with the parameter λ , so that $n^\mu = e_\lambda^\mu$, we make sure of generating the null hypersurface Σ by the geodesic integral curves of the future pointing null vector field $\frac{\partial}{\partial\lambda}$. We may then complete the basis by a transverse null vector N^μ uniquely defined by the four conditions $n_\mu N^\mu = -1$, $N_\mu e_A^\mu = 0$ ($A = \theta, \varphi$), and $N_\mu N^\mu = 0$ [1]. We find

$$N_\mu|_- = \frac{1}{2}(\dot{R} + \sqrt{1+E}) \left(1, \frac{R'}{\sqrt{1+E}}, 0, 0 \right)|_\Sigma, \quad (11)$$

$$N_\mu|_+ = \left(\frac{-1}{2} \left(1 - \frac{2M(v)}{R} \right), 1, 0, 0 \right)|_\Sigma. \quad (12)$$

The final matching conditions are formulated in terms of the jump in the transverse extrinsic curvature. Using the definition $\mathcal{K}_{ab} = e_a^\mu e_b^\nu \nabla_\mu N_\nu$ [1], we now compute the components of the transverse extrinsic curvature tensor on both sides of Σ . Its non-vanishing components on the minus side are found as

$$\mathcal{K}_{\theta\theta}|_- = \sin^{-2} \theta \mathcal{K}_{\varphi\varphi}|_- = \frac{R}{2} \left(1 - \frac{2M(r)}{R} \right)|_\Sigma, \quad (13)$$

$$\mathcal{K}_{\lambda\lambda}|_- = \frac{-M'(r)}{RR'(\dot{R} + \sqrt{1+E})^2}|_\Sigma. \quad (14)$$

The corresponding non-vanishing components on the plus side are

$$\mathcal{K}_{\theta\theta}|_+ = \sin^{-2} \theta \mathcal{K}_{\varphi\varphi}|_+ = \frac{R}{2} \left(1 - \frac{2M(v)}{R} \right)|_\Sigma. \quad (15)$$

Now, the requirement for the continuity of the different components of the transverse extrinsic curvature tensor across the null hypersurface Σ gives

$$M(r) \overset{\Sigma}{=} M(v), \quad (16)$$

$$\frac{M'(r)}{R'(\dot{R} + \sqrt{1+E})^2}|_\Sigma = 0, \quad (17)$$

where we have used Eqs. (13-15). The condition (16) simply expresses that the total gravitational mass as seen from the exterior must coincide with that seen from the interior on the

hypersurface, as would be expected. Furthermore, from the condition (17) we see that continuity of the $\lambda\lambda$ component of the transverse curvature tensor over Σ requires that one of the following cases be satisfied:

(i) The function $E(r)$ attains the unlimited value (i.e., $E(r) \rightarrow \infty$) on the hypersurface Σ . Given the following limiting forms on the hypersurface in a LTB collapsing model as studied in [6]

$$\begin{aligned} R &\rightarrow \sqrt{E}(a-t), \\ \dot{R} &\rightarrow -\sqrt{E}, \\ R' &\rightarrow \frac{RE'}{2E} + a'\sqrt{E}, \end{aligned} \tag{18}$$

where $a(r)$ is an arbitrary function, we can see that the denominator in the matching condition (17) goes to infinity so as to for the nonzero values of the numerator the condition (17) will be satisfied.

(ii) $M'(r)$ approaches zero on the hypersurface Σ while the function $E(r)$ remains finite.

In both cases (i) and (ii) from Eq. (3) we see that the energy density of LTB interior region ρ_L must go to zero on the hypersurface Σ , otherwise there will be a nonzero isotropic surface pressure given by [1]

$$p = -[\mathcal{K}_{\lambda\lambda}] = \frac{-4\pi R\rho_L}{(\dot{R} + \sqrt{1+E})^2}|_{\Sigma}, \tag{19}$$

signaling the presence of a thin distribution of matter on the hypersurface Σ with the surface quantity (19) which would be better described as a tension.

In the case of smooth matching of the two solution the jump in the component $T_{\mu\nu}n^\mu n^\nu$ of the energy momentum tensor is zero and in addition, it follows that λ is an affine parameter on the both sides of Σ .

The Kretschmann scalar $K = R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho}$ is computed for the LTB spacetime as [6]

$$K|_- = \frac{48M^2}{R^6} - \frac{32MM'}{R^5R'} + \frac{12M'^2}{R^4R'^2}, \tag{20}$$

and for the Vaidya spacetime

$$K|_+ = \frac{48M^2(v)}{R^6}. \tag{21}$$

By virtue of the matching conditions (16) and (17) together with the limiting form $R' \rightarrow \infty$ related to the case (i) one can easily show the continuity of the Kretschmann scalar over the null hypersurface for the both cases (i) and (ii).

3 Conclusion

Applying the Barrabès-Israel null shell formalism we have examined the matching of a LTB interior region to the Vaidya exterior spacetime along a null hypersurface. We have shown that in the context of BI formalism the limit of interest in which $E(r)$ diverges on the null hypersurface as considered in [6] stems from the requirement for the continuity of the components of the

second fundamental form across the hypersurface as needed for the smooth matching of the two spacetimes. Furthermore, we conclude that one can glue these two solutions to the Einstein equations smoothly through a null hypersurface even for the finite values of the function $E(r)$ provided that the function $M'(r) \rightarrow 0$ on Σ . In any case, the energy density of LTB interior region ρ_L goes to zero on the hypersurface Σ .

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